# Continuous $C^*$-bundles with fibres $\mathcal{O}_\infty$

E. Blanchard (CNRS, Paris)

# Cuntz C*-algebras

**Definition** (Cuntz)

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$$\mathcal{O}_\infty = C^* < s_1, s_2, s_3, \ldots; 1 = s_m^* s_m = \sum_m s_m s_m^* >$$

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**Definition** A non-zero unital C*-algebra $D$ is **$K_1$-injective** if any unitary $v \in \mathcal{U}(D)$ with $[v] = [1_D]$ in $K_1(D)$ satisfies $v \sim_h 1_D$

# Examples of Strongly Self-Absorbing C*-algebras

– The Cuntz algebra $\mathcal{O}_2 = C^* < s_1, s_2\,;\, 1 = s_1^* s_1 = s_2^* s_2 = \sum_{i=1,2} s_i s_i^* >$

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$$\mathcal{Z}_{p_n,q_n} = \{f \in C([0,1]; M_{p_n}(\mathbb{C}) \otimes M_{q_n}(\mathbb{C}) \,; \\ f(0) \in M_{p_n}(\mathbb{C}) \otimes \mathbb{C} \text{ and } f(1) \in \mathbb{C} \otimes M_{q_n}(\mathbb{C})\}$$

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is a unital C$^*$-subalgebra $\boxed{A \subset \prod_{x \in \mathbf{X}} A_x}$ such that:

(a) There is a unital $*$-embedding $C(\mathbf{X}) \to A$ given by

$$f \mapsto (f(x)1_{A_x}) \text{ for all } f \in C(\mathbf{X})$$

(b) For all $x \in \mathbf{X}$, the fibre map $A \to A_x$ is surjective.

(c) $\forall\ (a_x)_{x \in \mathbf{X}} \in A, \quad \mathbf{x} \mapsto \|\mathbf{a_x}\|_{\mathbf{A_x}}$ is $\boxed{\text{continuous.}}$

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Thus, $D \not\cong C(\mathfrak{X}; \mathcal{O}_2)$.

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**Proposition** (B., Kirchberg)
$\mathcal{T}_{C(\mathfrak{X})}(E)$ is a unital continuous $C(\mathfrak{X})$-algebra
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Hence $\mathcal{T}_{C(\mathfrak{X})}(E)$ is a **locally purely infinite** C*-algebra.

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– But the functor $\varprojlim_{k \in \mathbb{N}^*}$ is not continuous...

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b) $\mathcal{T}_{C(\mathfrak{X})}(E)$ purely infinite C$^*$-algebra.

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c) $\forall x \in \mathfrak{X}$, $\exists x \in \overset{o}{F}(x) \subset F(x)$ with $\mathcal{T}_{C(\mathfrak{X})}(E)_{|F(x)}$ purely infinite.

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**Question:** Is $\mathcal{T}_{C(\mathfrak{X})}(E) \cong \mathcal{T}_{C(\mathfrak{X})}(E) \otimes \mathcal{O}_\infty$?

**Proposition** (B., Dădărlat, Kirchberg, Rørdam, Toms, Winter)
The following assertions are equivalent:

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$\rightsquigarrow$ local problem

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**Substance of the Proof.**

$$u_m = \sum_{k=1}^{n} (\phi_k)^{1/2}.\ell(0 \oplus \zeta_k)^{mn+k}.L \qquad \text{isometry in } \mathcal{T}_{C(\mathfrak{X})}(C(\mathfrak{X}) \oplus E)$$

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# Local and Global Pure Infiniteness

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**Proof.** $\mathcal{T}(E_x) \cong \mathcal{O}_\infty$ $\boxed{\text{semiprojective}}$.

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But this cannot be.

# End of the proof

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And so $\mathcal{O}_\infty \not\hookrightarrow \mathscr{L}_{C(\mathfrak{X})}(\mathcal{E})$

# Concluding remark

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– Some quotient of $\mathcal{T}_{C(\mathfrak{X})}(\mathcal{E})$ is a properly infinite $C^*$-algebra which is not $K_1$-injective.